Brief notes on non-linear transformations to bound conductivity during EM inversion

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March 21, 2011

Bounding model parameters

Occam's inversion seeks to minimize the functional

$$U = \|\boldsymbol{\partial}\mathbf{m}\|^2 + \mu^{-1}\left[\|\mathbf{W}((\mathbf{d}) - \mathcal{F}(\mathbf{m}))\|^2 - \chi_*^2\right], \tag{1}$$

where the first term is the model roughness and the second term is the fit of model $\mathbf{m}$'s forward response $\mathcal{F}(\mathbf{m})$ to the data $\mathbf{d}$. This equation is linearized about a starting model $\mathbf{m}_k$ and the minimization is accomplished iteratively using:

$$\mathbf{m}_{k+1} = \left[\mu\left(\boldsymbol{\partial}^T\boldsymbol{\partial}\right) + (\mathbf{W}\mathbf{J}_k)^T\mathbf{W}\mathbf{J}_k\right]^{-1}\left[(\mathbf{W}\mathbf{J}_k)^T\mathbf{W}\hat{\mathbf{d}}\right] \tag{2}$$

where

$$\hat{\mathbf{d}} = \mathbf{d} - F(\mathbf{m}_k) + \mathbf{J}_k\mathbf{m}_k. \tag{3}$$

The model Jacobian matrix is

$$J_{ij} = \frac{\partial F_i(\mathbf{m}_k)}{\partial m_j} = \frac{\partial F_i(\mathbf{m}_k)}{\partial \log_{10}\sigma_j}. \tag{4}$$

In other words, the linearized sensitivity of the i-th datum to the j-th parameter. Now suppose we'd like to enforce bounds on the model parameters, such that

$$b < m < a \tag{5}$$

We can accomplish this by inverting for a new model parameter $\mathbf{x}$, where $-\infty < x(m) < \infty$. The bounds are enforce by designing the transform such that the inverse transform $m(x)$ is bounded. Before going on the specific transforms, the modified model update equation is then

$$\mathbf{x}_{k+1} = \left[\mu\left(\boldsymbol{\partial}^T\boldsymbol{\partial}\right) + (\mathbf{W}\mathbf{J}_k^*)^T\mathbf{W}\mathbf{J}_k^*\right]^{-1}\left[(\mathbf{W}\mathbf{J}_k^*)^T\mathbf{W}\hat{\mathbf{d}}\right] \tag{6}$$

where
$$\hat{\mathbf{d}} = \mathbf{d} - F(\mathbf{m}(\mathbf{x})_k) + \mathbf{J}_k^* \mathbf{x}_k. \tag{7}$$
and
$$J_{ij}^* = \frac{\partial m_j}{\partial x_j} J_{ij} \tag{8}$$

As you should now see, we need to define the transform $x(m)$, its inverse $m(x)$ and the sensitivity scaling $\partial m_j / \partial x_j$.

Bounds with a simple exponential equation

As shown in Habashy and Abubakar (2004) and later in Commer and Newman (2008), a simple transformation to bound the parameters is given by:

$$m = \frac{ae^x + b}{e^x + 1} \quad \text{when} \quad x \le 0 \tag{9}$$

$$m = \frac{a + be^{-x}}{1 + e^{-x}} \quad \text{when} \quad x < 0 \tag{10}$$

where the equations evaluate to the same value, but are broken up so that the exponential functions are always decaying, and hence will be numerically safer to program (ie., avoiding large positive exponents). We can see that for large positive x, $m(x)$ asymptotes to a, while for large negative x, $m(x)$ asymptotes to b. Taking the derivative of these, we find the sensitivity scaling to be:

$$\frac{\partial m}{\partial x} = \frac{(a - b)e^x}{(1 + e^x)^2} \quad \text{when} \quad x \le 0 \tag{11}$$

$$\frac{\partial m}{\partial x} = \frac{(a - b)e^{-x}}{(1 + e^{-x})^2} \quad \text{when} \quad x > 0. \tag{12}$$

The transform $x(m)$ can be found from the first equation to be:

$$x = \log(m - b) - \log(a - m), \quad b < m < a \tag{13}$$

Figure 1 shows an example of the transform (red curve) and Figure 2 shows the sensitivity scaling. I found the peak in the scaling curve to be curious, and wondered if that might bias the inversion towards the value $(a+b)/2$, at least as much as it could while still fitting the data. So that led to...

Bounds with a flat sensitivity response: the bandpass transformation

It seemed to me that a flat sensitivity response might be better, and that reminded me of the formula for a band-pass filter. The band-pass filter equation is just

$$\frac{\partial m}{\partial x} = \frac{1}{(1 + e^{-c(x-b)})(1 + e^{c(x-a)})} \tag{14}$$

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or in numerically convenient forms:

$$\frac{\partial m}{\partial x} = \frac{e^{c(x-b)}}{(1 + e^{c(x-b)})(1 + e^{c(x-a)})} \quad \text{when} \quad x \le 0 \tag{15}$$

$$= \frac{e^{-c(x-a)}}{(1 + e^{-c(x-b)})(1 + e^{-c(x-a)})} \quad \text{when} \quad x > 0 \tag{16}$$

Empirical test show that $c = d/(a - b)$ with $d = 15$ works well for giving the steep response corners for typical conductivity bounds. The $(a - b)$ term makes it scale with the separation of the bounds. The corresponding equation for $m(x)$ can be found by integrating this equation:

$$m(x) = \frac{c(b+a) + \log(1 + e^{c(x-b)}) - \log(1 + e^{c(x-a)})}{c(1 - e^{c(b-a)})} - a \tag{17}$$

with alternative solutions

$$m(x) = \frac{1}{c(1 - e^{-c(a-b)})} \log\left(\frac{1 + e^{c(b-x)}}{e^{-ac}(1 + e^{c(a-x)})}\right) \tag{18}$$

$$= \frac{ac + \log\left(\frac{1 + e^{c(b-x)}}{1 + e^{c(a-x)}}\right)}{c(1 - e^{-c(a-b)})} \quad \text{when} \quad x > 0 \tag{19}$$

$$= \frac{ac + \log\left(\frac{e^{cx} + e^{cb}}{e^{cx} + e^{ca}}\right)}{c(1 - e^{-c(a-b)})} \quad \text{when} \quad x \le 0 \tag{20}$$

$$\tag{21}$$

Solving for $x(m)$ yields:

$$x(m) = \frac{1}{c} \log\left(\frac{e^{ac}(e^{\tilde{m}} - e^{bc})}{e^{ac} - e^{\tilde{m}}}\right) \tag{22}$$

$$\text{where} \quad \tilde{m} = mc(1 - e^{-c(a-b)}) \tag{23}$$

Figure 2 shows the flat sensitivity response of this transform between in the pass-band. Figure 1 (blue curve) shows $m(x)$. Notice how the transformed parameters are identical to the original parameters between the bounds.

Both of these transforms have been implemented in Occam1DCSEM. My initial tests show them both to work well. My suspicion that the exponential transform might bias the inversion to $(a+b)/2$ does not seem to be the case. In terms of performance, for some data sets the band-pass transform leads to quicker convergence, while for other data sets the exponential transform converges faster; so far I find no reason to prefer one or the other.
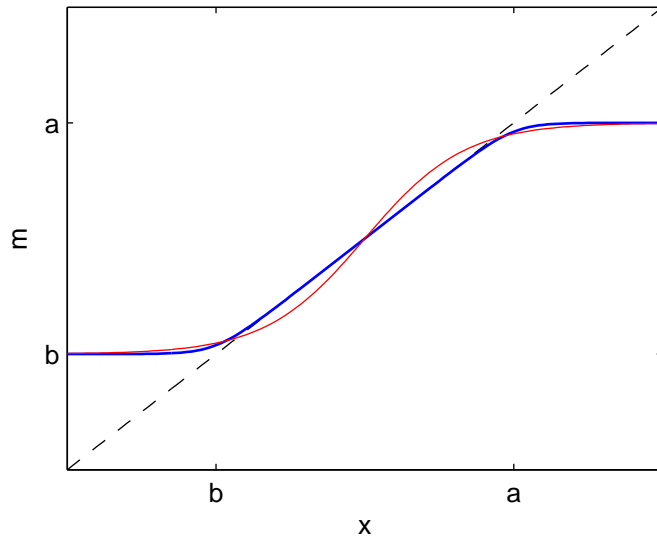
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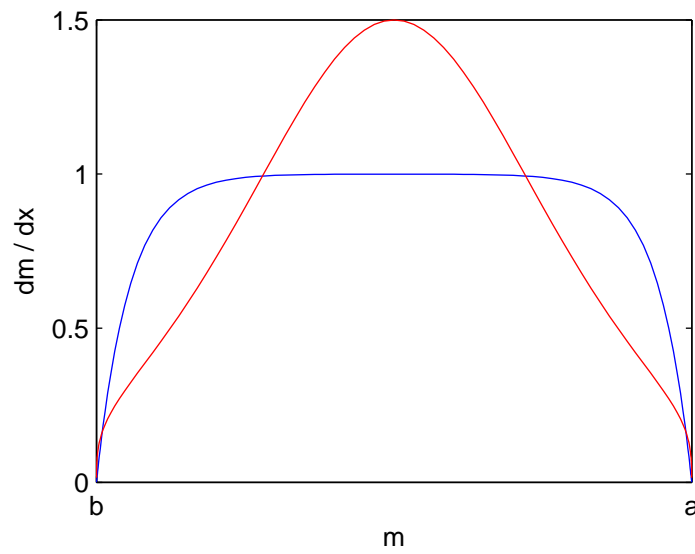Figure 1: $m(x)$ shown for the exponential transform (red) and the band-pass transform (blue).

Figure 2: The sensitivity scaling $\frac{\partial m}{\partial x}$ shown for the exponential transform (red) and the band-pass transform (blue).

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References

Commer, M. and G. A. Newman, 2008, New advances in three-dimensional controlled-source electromagnetic inversion: Geophysical Journal International, **172**, 513–535.

Habashy, T. and A. Abubakar, 2004, A general framework for constraint minimization for the inversion of electromagnetic measurements: Progress in electromagnetics Research, **46**, 265–312.